Technical Comments

Comment on "Evolution of the Laminar Wake behind a Flat Plate and Its Upstream Influence"

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DURING the last few years considerable effort has been expended in an attempt to understand the character of boundary-layer flow at a sharp trailing edge. The Introduction to Ref. 1 summarizes most of the published investigations of steady incompressible laminar flow past the trailing edge of a finite plate at zero incidence and high Reynolds number. In particular, asymptotic descriptions considered to be valid in the limit as $R \to \infty$ ($R = UL/\nu$; U = external flow speed, L = plate length, ν = kinematic viscosity) were given independently by Stewartson[2] and by Messiter.[3] For the most part, Ref. 1 correctly points out the deficiencies of earlier work, but several comments about the theory derived by Stewartson and Messiter are either unjustified or incorrect.

First of all, Ref. 1 states incorrectly that the work of Ref. 4 assumed isobaric flow. The assumptions actually made were that: a) the effect of the trailing edge is confined to a distance $O(R^{-3/4}L)$; and b) the Navier-Stokes equations can be linearized about the Blasius solution in this neighborhood. The serious error lay in neglecting the discontinuity of streamline slopes in the boundary layer at $x = 1$; this was corrected in Refs. 2 and 3. Imai treated the same problem as in Ref. 4 but with the isobaric assumption. Incidentally, Ref. 4 is not without value for it still provides a useful guide to the structure of the region within a distance $O(R^{-3/4}L)$ of the trailing edge provided one adjusts the constant λ as explained in Ref. 2. Hakkinen and O'Neil,[5] not mentioned in Ref. 1, have also discussed this region, in terms of expansions for $R^{3/4}r/L \to \infty$.

Second, it is stated in Ref. 1 that the use of linearized airfoil theory is not justified because of the large slopes of the (effective) slender body. It is certainly true that the slopes are large compared with the streamline slopes predicted by ordinary boundary-layer theory, but the slopes do remain small compared to 1. As explained in Refs. 2 and 3, $u = 1 + O(R^{-1/4})$, $v = O(R^{-1/4})$, $\partial u/\partial x = O(R^{+1/8})$, etc. In fact, Eq. (7) of Ref. 1 essentially incorporates the notion of linearization because the term $v\,\partial u/\partial y$ is omitted. Also, as Fig. 6 of Ref. 1 makes clear, $|c_p| \sim 0.01$ and therefore $c_p = -2(u-1)$ with error ~ 0.0001.

Third, the main problem studied in Eqs. (4–7) of Ref. 1 contains the formulation of Refs. 2 and 3 and reduces to it as $R \to \infty$. Thus there is agreement in this limit. However, this does not provide justification for neglecting terms when $R = 10^5$. A number of terms are omitted in Ref. 1, but perhaps the most important is the pressure variation across the boundary layer. It is established in Refs. 2 and 3 that this is $O(R^{-3/8})$, just a fraction smaller than the longitudinal pressure variation $O(R^{-1/4})$. Neglecting this term is justified in the limit as $R \to \infty$, but may well lead to numerical inaccuracy at $R = 10^5$.

Fourth, the pressure plotted in Fig. 6 of Ref. 1 shows a singularity at $x = 1$, and so dp/dx is seriously in error if x is very close to 1. The solutions to Eqs. (1–3) of Ref. 1 presumably require that the pressure be known at "the edge of Region I"; however, the pertinent details are not discussed. Furthermore, the numerical work of Ref. 1 would have benefited from making use of an asymptotic analysis of the outer region of influence of the trailing edge. Conditions at infinity are notoriously difficult to handle numerically, and the discrepancy between the results of Ref. 1 and those of Goldstein, when $x = 2$, should be regarded with caution.

Finally, Fig. 6 of Ref. 1 is in qualitative agreement with the predictions of Refs. 2 and 3, in that a favorable pressure gradient is established for $x < 1$, an adverse pressure gradient at $x = 1^+$, and a favorable pressure gradient as $(x-1)R^{3/8} \to \infty$. Fig. 6 shows that the displacement effect leads to large dc_p/dx for $\Delta x \approx 0.2$ (say) whereas Fig. 10 shows significant changes in skin friction for $\Delta x \approx 0.0005$. While the dependence on R is not clearly established by these results, the curves do show unequivocally the existence of two length scales, again qualitatively in agreement with the results of Refs. 2 and 3. The latter references do not contend that the region $O(R^{-3/4}L)$ is irrelevant to the structure of the flow, but rather that the dominant correction occurs on the larger scale, and that even for $R = 10^5$ (say) the largest effects are found in a qualitatively correct way by the asymptotic analysis given. For example, the correction to the drag coefficient on the larger scale is $O(R^{-7/8})$, whereas effects on the smaller scale lead to a change $O(R^{-5/4})$. The correction may be calculated either by integrating skin friction along the plate or by calculating a momentum deficit in the wake. For consistency the eigenfunction proposed in Eq. 5.10 of Ref. 2 is required.

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Reply by Author to A. F. Messiter and K. Stewartson

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THE authors thank Professors Messiter and Stewartson for their interest in and critical evaluation of the results in Ref. 1. Unfortunately, a portion of their comment is based on an error in the analysis section of Ref. 1 wherein the term $v\,\partial u/\partial y$ was omitted from Eq. (7). As inferred implicitly in the discussion of the solution method,[1] the potential flow problem was solved "exactly".[2] Whether or not this was necessary is perhaps debat-

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